Journal Club QUFIBA

Ventaja cuántica en la simulación de procesos estocásticos.

Quantum Advantage in Simulating Stochastic Processes
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Phys. Rev. X 11, 021019 – Published 22 April 2021

Ventaja cuántica en la simulación de procesos estocásticos.

Temas a tratar:

- Motivación
- Embbedability of stochastic processes
- Space-time cost of a stochastic process
- Role of memory in state transformations

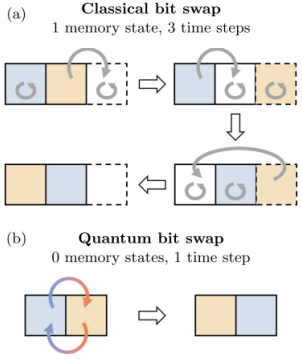
Motivación

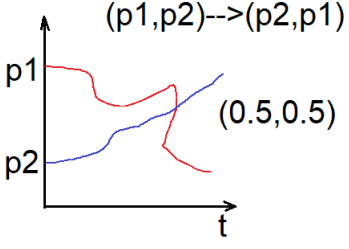
Sabemos que existen problemas cuya solución puede ser muy difícil de hallar con computadoras clásicas. Sin embargo, algunos de ellos pueden ser resueltos de manera mucho más sencilla efectuando operaciones sobre un sistema cuántico. En esto se basa la computación cuántica. Ejemplos de algotimos que presentan una ventaja cuántica son

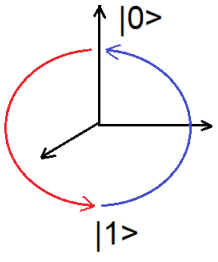
- Deutsch–Jozsa
- Shor
- Quantum Fourier transform

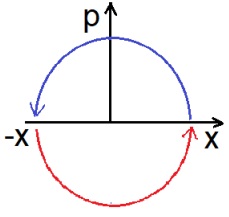
¿Será posible también aprovechar los sistemas cuánticos para computar más eficientemente procesos estocásticos?

Motivación


(a)
Classical bit swap
1 memory state, 3 time steps
(b)
Quantum bit swap
0 memory states, 1 time step


(p1,p2)-->(p2,p1)
p1
p2
(0.5,0.5)
t


|0>
|1>


p
-x
x

Embbedability of stochastic processes

¿Qué es un proceso estocástico? Dado un vector de probabilidad $\mathbf{p} \in [0,1]^d$, $\sum_{i=1}^{d} p_i = 1$

Un proceso o matriz estocástica es una matriz $P \in [0,1]^{d \times d}$ tal que

$$P_{i|j} \geq 0, \qquad \sum_i P_{i|j} = 1,$$

Entonces $\mathbf{q} = P\mathbf{p}$ es también un vector de probabilidad.

Nos interesan especialmente los procesos estocásticos que no requieren memoria (markovianos) ya que pueden ser implementados de manera especialmente eficiente.

Classical embbedability

Se dice que P es **embbedable** o que no requiere memoria si existe $P(t)$ tal que

$$\frac{d}{dt}P(t) = L(t)P(t), \qquad P(0) = 1.$$

y $P = P(t_f)$, siendo

$$L_{i|j} \geq 0 \quad \text{for } i \neq j, \qquad \sum_i L_{i|j} = 0,$$

Una condición necesaria para que una matriz P sea embbedable es que

$$\prod_i P_{i|i} \geq \det P \geq 0.$$

Embbedability of stochastic processes

Estado cuántico dado por un operador densidad ρ en un espacio de Hilbert d-dimensional $\mathcal{H}_d$ de base $\{|k\rangle\}_{1\leq k\leq d}$. Evoluciona temporalmente hasta el estado $\rho' = \varepsilon(\rho)$, ε es un canal cuántico (completely positive trace preserving map). Estado $\rho_{\mathrm{p}} = \sum_k p_k |k\rangle\langle k|$ aplicamos $\varepsilon(\rho_p)$, la probabilidad de encontrar al sistema en cada estado de la base viene dada por $P\mathbf{p}$ con

$$P_{i|j} = \langle i|\mathcal{E}(|j\rangle\langle j|)|i\rangle.$$

Un canal cuántico ε es markoviano si existe $\varepsilon(t)$ tal que

$$\frac{d}{dt}\hat{\mathcal{E}}(t) = \hat{\mathcal{L}}(t)\hat{\mathcal{E}}(t), \qquad \hat{\mathcal{E}}(0) = \hat{\mathcal{I}},$$

con $\varepsilon = \varepsilon(t_f)$, siendo

$$\mathcal{L}(\cdot) = -i[H,\cdot] + \Phi(\cdot) - \frac{1}{2}\{\Phi^*(\mathbb{1}),\cdot\},$$

Quantum embbedability

Una matriz estocástica P es quantum embbedable si existe un canal markoviano cuántico ε tal que

$$P_{i|j} = \langle i|\mathcal{E}(|j\rangle\langle j|)|i\rangle,$$

No todas las quantum embbedable son classically embbedable

$$P = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} \quad U = \begin{bmatrix} \sqrt{1/3} & \sqrt{2/3} \\ \sqrt{2/3} & -\sqrt{1/3} \end{bmatrix}$$

P es quantum embbedable con $\epsilon(.) = U^\dagger(.)U$ pero $\det P < 0$ no es clasically embbedable.

Embbedability of stochastic processes

Todas las matrices embbedable son quantum embbedable.

Dada una matriz embbedable P con generador L, basta definir a $\mathcal{L}$ como

$$\Phi(\cdot) = \sum_{ij} K_{ij}(\cdot)K_{ij}^{\dagger}, \qquad K_{ij} = \sqrt{L_{i|j}}|i\rangle\langle j|.$$

No vale la inversa! Las matrices de permutación Π satisfacen

$$\det \Pi = \pm 1, \qquad \prod_i \Pi_{i|i} = 0.$$

Con lo cual no verifican la condición de embbedable. Sin embargo, son unitarias y entonces pueden ser generadas por un hamiltoniano lo que las hace quantum embbedable.

Embbedability of stochastic processes

Lemma 1 (Monoid property). The set of quantum-embeddable stochastic matrices contains identity and is closed under composition; i.e., if P and Q are quantum embeddable, then PQ is also.

Corollary 2. All 2×2 stochastic matrices are quantum embeddable.

Es decir que cualquier proceso estocástico (de 2x2) por más memoria que requiera clásicamente puede ser procesado cuánticamente con un qubit sin ninguna memoria adicional!

Space-time cost of a stochastic process

Classical space-time cost

Definition 2 (Space cost). The space cost of a $d \times d$ stochastic matrix P, denoted $C_{\text{space}}(P)$, is the minimum m such that a $(d+m) \times (d+m)$ embeddable matrix Q implements P.

Definition 3 (One-step process). A stochastic matrix T is called a one-step process if

(1) it is embeddable;

(2) the controls $L(t)$ that generate T at time t_f through Eq. (3) can be chosen such that the set of nonzero transition probabilities of $P(t)$ is the same for all $t \in (0, t_f)$.

Definition 4 (Time cost). The time cost $C_{\text{time}}(P, m)$ of a $d \times d$ stochastic matrix P, while allowing for m memory states, is the minimum number τ of one-step stochastic matrices $T^{(i)}$ of dimension $(d+m) \times (d+m)$ such that $Q = T^{(\tau)} \cdots T^{(1)}$ implements P.

Space-time cost of a stochastic process

Quantum space-time cost

Definition 5 (Quantum space cost). The quantum space cost of a $d \times d$ stochastic matrix P, denoted $Q_{\text{space}}(P)$, is the minimum m such that the $(d+m) \times (d+m)$ quantum-embeddable matrix Q implements P.

Definition 6 (Quantum time cost) The quantum time cost $Q_{\text{time}}(P, m)$ of a $d \times d$ stochastic matrix P, while allowing for m memory states, is the minimum τ such that there exist time-independent Lindbladians $\mathcal{L}_1, \ldots, \mathcal{L}_\tau$ on a $(d+m) \times (d+m)$-dimensional Hilbert space and

$$Q_{i|j} = \langle i | e^{\mathcal{L}^{(\tau)} t_\tau} \cdots e^{\mathcal{L}^{(1)} t_1} (|j\rangle\langle j|) | i \rangle \tag{24}$$

implements P.

Space-time cost of a stochastic process

Classical cost

Let P_f be a $\{0,1\}$-valued $d \times d$ stochastic matrix defined by a function $f : \mathbb{Z}_d \to \mathbb{Z}_d$. Let $\mathrm{fix}(f)$ be the number of fixed points of f, $|\mathrm{img}(f)|$ the dimension of the image of f, and $\mathrm{c}(f)$ the number of cycles of f, i.e., the number of distinct orbits of elements of $\{1, \ldots, d\}$ of the form $\{i, f(i), f(f(i)), \ldots, i\}$. Recently, the following result has been shown.

Theorem 3 (Classical cost of a function [10]). The time cost of a $\{0,1\}$-valued stochastic matrix P_f described by a function f is given by

$$C_{\mathrm{time}}(P_f, m) = \left\lceil \frac{m + d + \max[\mathrm{c}(f) - m, 0] - \mathrm{fix}(f)}{m + d - |\mathrm{img}(f)|} \right\rceil + b_f(m)$$
$$\geq \left\lceil \frac{m + d - \mathrm{fix}(f)}{m + d - |\mathrm{img}(f)|} \right\rceil, \qquad (25)$$

where $b_f(m) = 0$ or 1 and $\lceil \cdot \rceil$ is the ceiling function.

Ejemplo, espacio de cadenas de longitud s: $d = 2^s$, implementar la función $f_1(i) = i \oplus 1$: $C_{time}(P_{f_1}, m) \geq 2^s/m$ es exponencial en la longitud de la cadena. Cuánticamente es una permutación que es quantum embbedable.

Space-time cost of a stochastic process

Quantum advantage

Theorem 4 (Quantum cost of a function). For any $m \geq 0$ and any function f, we have $Q_{\text{time}}(P_f, m) \leq 2$.

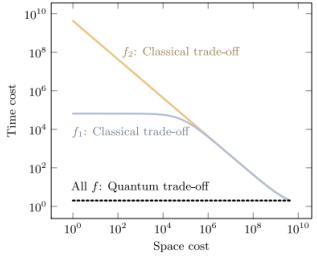


FIG. 5. Classical versus quantum space-time trade-off. The optimal trade-off between space cost and time cost of implementing stochastic matrices for a system of $s = 32$ bits, i.e., with dimension $d = 2^{32}$ (plotted in log-log scale). Solid colored curves correspond to optimal trade-offs for classically implementing exemplary $\{0, 1\}$-valued stochastic matrices described by functions $f_1(i) = i \oplus 1$ (addition modulo d) and $f_2(i) = \min\{i + 2^{s/2}, 2^s - 1\}$, as analyzed in Ref. [10]. The dashed black curve corresponds to optimal trade-offs for quantumly implementing any $\{0, 1\}$-valued stochastic matrix, thus illustrating a quantum advantage.

Role of memory in state transformations

Accesibility regions

Classical

Definition 7 (Classical accessibility). A distribution $\boldsymbol{q}$ is accessible from $\boldsymbol{p}$ by a classical stochastic process with a fixed point $\boldsymbol{\gamma}$ if there exists a stochastic matrix P, such that

$$P\boldsymbol{p} = \boldsymbol{q} \quad \text{and} \quad P\boldsymbol{\gamma} = \boldsymbol{\gamma}. \tag{27}$$

We denote the set of all $\boldsymbol{q}$ accessible from $\boldsymbol{p}$ given $\boldsymbol{\gamma}$ by $\mathcal{C}_{\gamma}^{\mathrm{Mem}}(\boldsymbol{p})$.

Por ejemplo, si P es un proceso de termalización el punto fijo γ será el estado térmico dado por

$$\gamma_k := \frac{1}{Z} e^{-\beta E_k}, \qquad Z := \sum_{k=1}^{d} e^{-\beta E_k}.$$

Definition 8 (Classical memoryless accessibility). A distribution $\boldsymbol{q}$ is accessible from $\boldsymbol{p}$ by a classical master equation with a fixed point $\boldsymbol{\gamma}$ if there exists a continuous one-parameter family $L(t)$ of rate matrices generating a family of stochastic matrices $P(t)$, such that

$$P(t_f)\boldsymbol{p} = \boldsymbol{q}, \qquad L(t)\boldsymbol{\gamma} = \boldsymbol{0} \quad \text{for all } t \in [0, t_f). \tag{29}$$

We denote the set of all $\boldsymbol{q}$ accessible from $\boldsymbol{p}$ given $\boldsymbol{\gamma}$ by $\mathcal{C}_{\gamma}(\boldsymbol{p})$.

Role of memory in state transformations

Quantum

Definition 9 (Quantum accessibility). A distribution $\boldsymbol{q}$ is accessible from $\boldsymbol{p}$ by quantum dynamics with a fixed point $\boldsymbol{\gamma}$ if there exists a quantum channel $\mathcal{E}$, such that

$$\mathcal{E}(\rho_{\boldsymbol{p}}) = \rho_{\boldsymbol{q}} \quad \text{and} \quad \mathcal{E}(\rho_{\boldsymbol{\gamma}}) = \rho_{\boldsymbol{\gamma}}. \tag{31}$$

We denote the set of all $\boldsymbol{q}$ accessible from $\boldsymbol{p}$ given $\boldsymbol{\gamma}$ by $\mathcal{Q}_{\boldsymbol{\gamma}}^{\mathrm{Mem}}(\boldsymbol{p})$.

Definition 10. [Quantum memoryless accessibility] A distribution $\boldsymbol{q}$ is accessible from $\boldsymbol{p}$ by a quantum master equation with a fixed point $\boldsymbol{\gamma}$ if there exists a continuous one-parameter family of Lindbladians $\mathcal{L}(t)$ generating a family of quantum channels $\mathcal{E}(t)$, such that

$$\mathcal{E}(t_f)[\rho_{\boldsymbol{p}}] = \rho_{\boldsymbol{q}}, \quad \mathcal{L}(t)[\rho_{\boldsymbol{\gamma}}] = 0 \ \text{ for all } t \in [0, t_f). \tag{32}$$

We denote the set of all $\boldsymbol{q}$ accessible from $\boldsymbol{p}$ given $\boldsymbol{\gamma}$ by $\mathcal{Q}_{\boldsymbol{\gamma}}(\boldsymbol{p})$.

Role of memory in state transformations

¿Ventaja cuántica?

$$\mathcal{Q}_{\gamma}^{\mathrm{Mem}}(\boldsymbol{p}) = \mathcal{C}_{\gamma}^{\mathrm{Mem}}(\boldsymbol{p}).$$

Theorem 5 (Maximal quantum advantage for uniform fixed points). For every $\boldsymbol{p}$ and a uniform distribution $\boldsymbol{\eta} = (1/d, \ldots, 1/d)$, one has $\mathcal{Q}_{\boldsymbol{\eta}}(\boldsymbol{p}) = \mathcal{C}_{\boldsymbol{\eta}}^{\mathrm{Mem}}(\boldsymbol{p})$.

Hay una gra ventaja a temperatura infinita!

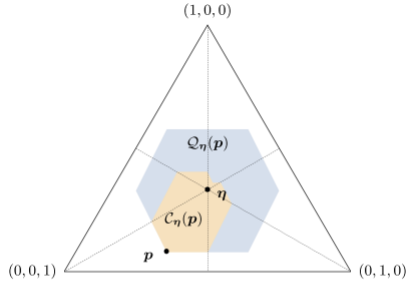


FIG. 6. Quantum advantage at infinite temperature for $d = 3$. We

Role of memory in state transformations

Qubit a temperatura finita

Let us start by recalling the classical solution to this problem. First, $\mathcal{C}_{\eta}^{\mathrm{Mem}}(\boldsymbol{p})$ can be obtained using the thermomajorization condition [38,40]. If $\boldsymbol{p} = (p, 1-p)$, in terms of the achievable ground-state populations, we have

$$\mathcal{C}_{\gamma}^{\mathrm{Mem}}(\boldsymbol{p}) = \begin{cases} [p, 1 - e^{-\beta E} p] & \text{if } p \leq 1/Z \\ [1 - e^{-\beta E} p, p] & \text{if } p > 1/Z. \end{cases} \tag{39}$$

Atraviesa el punto de equilibrio, information back-flow debido a la memoria.

$$\mathcal{C}_{\gamma}(\boldsymbol{p}) = \begin{cases} [p, 1/Z] & \text{if } p \leq 1/Z \\ [1/Z, p] & \text{if } p > 1/Z. \end{cases}$$

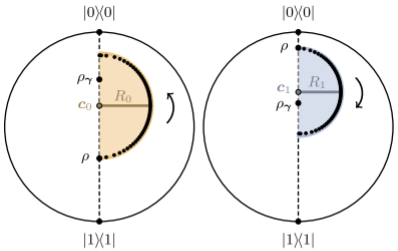
Role of memory in state transformations

Result 1 (Quantum advantage at every finite temperature—numerics). For $d = 2$, $\mathcal{Q}_\gamma(\boldsymbol{p}) = C_\gamma^{\mathrm{Mem}}(\boldsymbol{p})$.

This result showcases that the advantage of Theorem 5 is not limited to the special case of a uniform fixed point involving unitary dynamics. Superposition can substitute memory in the control of classical systems at every finite temperature.

In order to prove Result 1, we present an explicit construction and numerical evidence for an even stronger result.

Result 2. [Numerics] Every qubit state accessible via a qubit channel with a given fixed point can be achieved by a qubit Markovian master equation with the same fixed point.



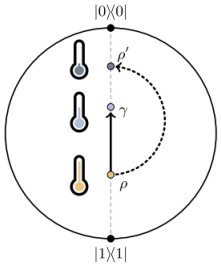


FIG. 3. Markovian cooling of a qubit. Classical memoryless

Role of memory in state transformations

PHYSICAL REVIEW X **11,** 011046 (2021)

Autonomous Temporal Probability Concentration: Clockworks and the Second Law of Thermodynamics

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(Received 13 July 2020; revised 26 November 2020; accepted 24 December 2020; published 8 March 2021)

According to thermodynamics, the inevitable increase of entropy allows the past to be distinguished from the future. From this perspective, any clock must incorporate an irreversible process that allows this flow of entropy to be tracked. In addition, an integral part of a clock is a clockwork, that is, a system whose purpose is to temporally concentrate the irreversible events that drive this entropic flow, thereby increasing the accuracy of the resulting clock ticks compared to counting purely random equilibration events. In this article, we formalize the task of *autonomous temporal probability concentration* as the inherent goal of any clockwork based on thermal gradients. Within this framework, we show that a perfect clockwork can be approximated arbitrarily well by increasing its complexity. Furthermore, we combine such an idealized clockwork model, comprised of many qubits, with an irreversible decay mechanism to showcase the ultimate thermodynamic limits to the measurement of time.

DOI: 10.1103/PhysRevX.11.011046 Subject Areas: Quantum Physics, Quantum Information

I. INTRODUCTION

that measures time inevitably requires an irreversible part that breaks this symmetry. By definition, the equilibrium

PHYSICAL REVIEW X **11,** 011047 (2021)

Improved Thermal Area Law and Quasilinear Time Algorithm for Quantum Gibbs States

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(Received 18 August 2020; revised 16 November 2020; accepted 22 December 2020; published 9 March 2021)

One of the most fundamental problems in quantum many-body physics is the characterization of correlations among thermal states. Of particular relevance is the thermal area law, which justifies the tensor network approximations to thermal states with a bond dimension growing polynomially with the system size. In the regime of sufficiently low temperatures, which is crucially important for practical applications, the existing techniques do not yield optimal bounds. Here, we propose a new thermal area law that holds for generic many-body systems on lattices. We improve the temperature dependence from the original $\mathcal{O}(\beta)$ to $\mathcal{O}(\beta^{2/3})$ up to a logarithmic factor, thereby suggesting subballistic propagation of entanglement by imaginary-time evolution. This qualitatively differs from the real-time evolution, which usually induces linear growth of entanglement. We also prove analogous bounds for the Rényi entanglement of purification and the entanglement of formation. Our analysis is based on a polynomial approximation to the exponential function which provides a relationship between the imaginary-time evolution and random walks. Moreover, for one-dimensional (1D) systems with n spins, we prove that the Gibbs state is well approximated by a matrix product operator with a sublinear bond dimension for $\beta = o(\log(n))$. This proof allows us to rigorously establish, for the first time, a quasilinear time classical algorithm for constructing a matrix product state representation of 1D quantum Gibbs states at arbitrary temperatures of $\beta = o(\log(n))$. Our new technical ingredient is a block decomposition of the Gibbs state that bears a resemblance to the decomposition of real-time evolution given by Haah *et al.* [*Proceedings of the 2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS)* (IEEE, New York, 2018), pp. 350–360].

DOI: 10.1103/PhysRevX.11.011047 Subject Areas: Condensed Matter Physics, Quantum Physics, Quantum Information

Fundamental Energy Requirement of Reversible Quantum Operations

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(Received 15 November 2019; revised 14 January 2021; accepted 3 March 2021; published 15 April 2021)

Landauer's principle asserts that any computation has an unavoidable energy cost that grows proportionally to its degree of logical irreversibility. But even a logically reversible operation, when run on a physical processor that operates on different energy levels, requires energy. Here we quantify this energy requirement, providing upper and lower bounds that coincide up to a constant factor. We derive these bounds from a general quantum resource-theoretic argument, which implies that the initial resource requirement for implementing a unitary operation within an error ϵ grows like $1/\sqrt{\epsilon}$ times the amount of resource generated by the operation. Applying these results to quantum circuits, we find that their energy requirement can, by an appropriate design, be made independent of their time complexity.